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VALUING MATHEMATICAL ROUTINES FOR SUPPORTING DIVERSE LEARNERS' MEANINGFUL ENGAGEMENT WITH MATHEMATICAL IDEAS AND MATHEMATICAL COMMUNICATION

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Abstract: *In this paper I highlight the importance of using a variety of short-duration mathematical routines, such as notice-wonder, which one doesn't belong, number talks, pattern talks, and quick builds to develop elementary and early secondary students' numerical reasoning, algebraic reasoning, and spatial thinking. These routines provide a foundation skillset for more advanced mathematical reasoning, including the abilities to justify, communicate clearly and effectively, and appreciate that problems can have more than one solution. I will illustrate several examples that connect a mathematical routine with a rich meaningful problem drawn from number sense and patterning and algebra. Most importantly, though, I argue that routines are essential in building a healthy mathematics classroom culture and meaningful relationships.*

Keywords: *routines, communication, spatial thinking, algebraic reasoning*

INTRODUCTION

Mathematical Routines

A *routine* is a sequence of actions that is performed on a regular basis as opposed to only on special occasions or in special places; the latter two would constitute *ritual* and c*eremony* respectively. Over time, routines can develop into *habits* of thinking, knowing and doing. Berry (2018) writes that routines are essential in mathematics education because "they give structure to time and interactions, letting students know what to expect in terms of participation, supporting classroom management and organization, and promoting productive classroom relationships for teaching and learning" (blog). Routines can be a *process* (instructional routine, management routine, evacuation routine) as well as a type of mathematical *task description*, that is, a routine task. In this paper I use routine in both senses. If the meaning is likely to be unclear, I will state whether I am referring to a task or process.

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There have been many studies of classroom routines in mathematics education over time (Lavie et al., 2019; Leinhardt, 1989). These have illustrated the importance of routines (both process and task) for effective teaching and classroom management that enables more ambitious pedagogy such as those recommended in reform-oriented (in contrast to traditional) curriculum documents and resources. At present, there exists a growing set of print books (Franke et al., 2018; McRoy et al., 2013; Milou & SanGiovanni, 2019; Parrish, 2014; SanGiovanni & Milou, 2019; Shumway, 2011) as well as multi-media digital resources (https://tapintoteenminds.com/3act-math/) that support ambitious mathematics teaching (Lampert et al., 2010). There is also a rich scholarly conversation in academic (Greer & Erickson, 2019; Hanner et al., 2019; Lampert et al., 2010) and professional publications (Clark, 2017; Marin, 2018; Newell & Chase, 2018; Rumack & Huinker, 2019). These print and digital resources repeatedly demonstrate that instructional routines (process) and specific mathematical routines (task) are critical in realizing mathematics proficiency and equity for all learners.

More generally, stable, meaningful routines at home and in the family unit have been repeatedly shown to be a key determinant of child well-being in ecocultural theory (Weisner, 2014). However, a significant number of students arrive at school today without sufficient experiences of daily routines necessary for flourishing in school. A key factor in helping students succeed in school, then, is to provide them with access and sufficient opportunities to practice and develop this important form of cognitive capital in the relevant disciplinary and cross-disciplinary learning contexts. As the dictum goes, "we are what we repeatedly do" or, as Lemov et al. (2012) succinctly put it, "practice doesn't make perfect, practice makes permanent" (p.xii).

In terms of considerations of equity, while routines and predictability are important for all students, they are especially important for neurodiverse learners, learners with developmental delays, in mainstream classrooms, and learners who are developing socio-emotional competencies such as self-regulation and active listening in classroom settings. Routines and predictability provide a regular and repeated structure upon which these students can rely.

Routines are critical to building a healthy culture for learning in the classroom. In a recent synthesis of the literature, Cantor et al. (2018) identify the foundational knowledge across multiple disciplines of relevance to education of how human beings develop in contexts (i.e., learn). They zero in on features of relationships and contexts that are drivers and delimiters of human development and articulate the strong convergences among the sciences studying learning. These ideas are reiterated and elaborated upon in the most recent (2018) National Academy of Sciences Engineering and Medicine Consensus report, "How People Learn 2: Learners Cultures and Context", in which one key finding is that "the human relationship is a primary process through which biological and contextual factors mutually reinforce each other" (p.310).

The majority of routines (process) in mathematics education in the Caribbean, however, are related to classroom behavioural management (Khan, 2006). Within instructional situations, traditional teacher-led I-R-E (teacher-Initiates-student-Responds-teacher-Evaluates) communicative sequences in blocked practice routines are the norm with interleaved (or spaced) practice used infrequently or only on final summative assessments modelled after terminal examinations such as Caribbean Secondary Education Certificate (CSEC) or Caribbean Advanced Proficiency Examination (CAPE).

In terms of specific mathematics routines (task), choral counting continues to be an effective strategy for learning some foundational aspects of multiplication, such as skip counting and remainders modulo ten pattern noticing (Franke et al., 2018) but is perhaps not sufficiently developed into a deep conceptual understanding of multiplication. There is a need for additional empirical research in this area to determine the types, frequency, and effects of diverse mathematical routines in classrooms from early childhood through post-secondary education in Caribbean settings.

In my own experience as a primary and secondary school student in Trinidad and later as a secondary school teacher of mathematics, I do not recall using any mathematical routines as described in this paper on a regular basis. Consonant with research I conducted, my own classroom (as a student and teacher) was dominated by use of the IRE model, blocked practice, and correcting homework with little time given to student-prompted discussions of "why"? The purpose of this article is to demonstrate the value of a few powerful or high-leverage routines for supporting the learning of mathematics that I have used successfully with pre-service elementary teachers. The secondary purpose is to encourage readers to adapt, develop, and share as widely as possible their own routines with others.

Routines in Primary/Elementary Teacher Education

In this section I describe a core set of routines that I use with both pre-service and in-service elementary teachers. Feedback after practicum and from in-service teachers demonstrates that the routines are seen as high leverage for student learning engagement and communication.

<u>Routine: Which one doesn't belong?</u>

Teacher: We are going to look at an image together. The image will have four sections. In each section you'll see a different mathematical object – one in the upper left, one in the upper right, one in the lower left and one in the lower right [point to each section as named on a blank square grid drawn on the board or chart paper]. I'll show you those four objects and ask you to think to yourself which one seems different from the others, and why. I'll ask you not to say anything at first so that everyone has a chance to think. But then I'll call on someone to talk to us about which object they chose. Our goal will be to understand how that person is thinking. Then I'll call on someone else. The great thing about these objects is that you can pick any object and find a reason for it not to belong. Every object is like the others in some ways, and different from them in some ways. So, this isn't a game about trying to guess the right answer, about trying to figure out what the teacher was thinking. This is a game in which you try to see new things in these objects – through your own ideas and by listening to your classmates. Thumbs up if you are ready (Teaching introductory script adapted from Danielson, 2016).

Teacher: Ok. I can see you are ready. Which one doesn't belong? [Figure 1 is shown]

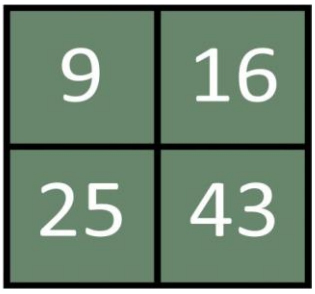

Figure 1
WODB Prompt. Image credit: Pam Wilson via http://wodb.ca/numbers.html (No. 1)

Student: I think 9 doesn't belong because it is the only single digit number.

Teacher: Does anyone have a different number they think doesn't belong?

Student: I think 16 doesn't belong because it is the only even number.

Student: I think 43 doesn't belong because it is the only number that is not a square number.

Teacher: What do you mean by square number?

Student: [Elaborating] 9 is 3 x 3, 16 is 4 x 4 and 25 is 5 x 5.

Student: I think 9 doesn't belong because it's the only one that is not 7.

Teacher: What do you mean by it is not seven?

Student: Well 1+ 6 = 7, 2 + 5 =7 and 4 + 3 = 7.

Teacher: Ok. We call that the digit sum. So the digit sum of the numbers is not seven.

Student: I think 25 because it is the only one with an even number.

Teacher: Are you sure? Have another look.

Student: Oh, right, 16 has an even number. I mean the only one with an even number in the tens position.

Teacher: Are you quite certain?

Student: Oh wait, 43 has an even number in the tens position too. I must have overlooked it. I can't think of a reason for 25 not to belong.

Teacher: Does anyone have a reason that 25 might not belong?

The preceding script demonstrates how the which-one-doesn't-belong (WODB) routine might be introduced using numbers at any point from the upper primary level onwards. The accompanying website *(wodb.ca)* or twitter hashtag #WODB as well as the books by Danielson (2016) provide many additional examples that can be used for different ages and topics. As you re-read the script and imaginatively put yourself in this classroom situation, think about what the students are doing and what the teacher is doing and how it might be different from what typically occurs in your classroom or in those of your peers.

The intent and enaction of the WODB routine are to shift the classroom discourse and conversation from one of correctness ("Am I right"?) to one of justification ("I think this is true because…"). This shift changes the learners' strategy in responding to a teacher prompt from *guessing* or attempting to read the teacher's mind or unstated intention to *reasoning* and attempting to express mathematical ideas precisely in order to communicate effectively and to convince others.

In terms of generic mathematical content, across multiple instantiations of WODB activities, learners are provided a low stakes opportunity to notice and make use of properties/attributes that can be visual, conceptual, or both. For instance, while the previous script draws on conceptual understanding related to numbers in most instances, the noticing of single digit versus double digits is at the visual discriminatory level first before the conceptual is explicitly called upon. Additional examples can be found at the *wodb.ca* website and in Danielson's (2016) book and blog. Readers are encouraged to explore these resources.

Additionally, learners have opportunities to think of critical features (Marton, 2014) and how things belong together and do not belong together (sorting and classifying) or to use definitions of mathematical objects/concepts in order to reason and justify. They are provided an opportunity to reason and justify and engage in rich mathematical communication (discourse) in the classroom with the locus of control for the evaluation of "rightness" shifting over time from the authority of the teacher and textbook to peers and to oneself in making sense of mathematical argumentation.

In terms of learning theory, working with the negation, the "not" or "exclusion" in these tasks, is cognitively effortful and contributes to students' productive struggle (Warshauer, 2015). In terms of classroom management, the task has a high ratio, that is, there is a high payoff for student learning along multiple dimensions of proficiency relative to the cost in terms of time and resources for teacher preparation. That the tasks are available for use free of charge (though a Creative Commons license signalling whether the online resources are true Open Education Resources [OER] would increase the clarity about the range of acceptable usages) in educational settings along with the large community of contributors online and on Twitter make it one that teachers can easily incorporate into lessons.

In cases where a projector is not available for classroom use, the grid can be drawn or each section printed out. The task provides a quick assessment (Fennell et al., 2017; Zager, 2018) and a way to easily support the highly effective and evidence-backed strategy of interleaving or non-blocked practice by bringing back content after a period of forgetting (Rohreh et al., 2019). This type of practice has been shown to increase connections and retention, and a quick routine that can be customized with different content such as WODB can facilitate this practice into regular pedagogy without the high stakes of assessment or "marks". Finally, although presented here in the context of mathematics education, the WODB routine is adaptable to any subject matter that involves the discernment of differences of attributes and does not have to be restricted only to visual stimuli.

Routine: The notice-wonder routine

In a YouTube video viewed over 20,000 times (The Math Forum, 2015), mathematics educator Annie Fetter provides an introduction to one of today's foundational routines in mathematics education – notice-wonder. In the video, Fetter describes two different situations in which she presented learners with a problem situation but first prompts them with the open question, "What do you notice"? In the classroom case she records the noticings. This is then followed by the second open question prompt, “What do you wonder”? and again the wonders are recorded for the whole class to see. Asking this second question, she argues, invites students to “display curiosity not misunderstanding” (02:50). Though misunderstandings (or what the literature might call misconceptions) do turn up, they are not negatively marked as such. Fetter goes on to illustrate how the routine “provides a platform for connecting student thinking and lets kids know that they have ideas in math that are valuable and useful” (04:19).

I often, if not always, begin work with a new group of teachers with this routine. From the very first session, the idea of noticing and honouring student wonderings is seeded and interleaved across their entire course of study. My intention is for the routine to form the foundation of a habit. Reading across the reflective electronic portfolios of pre-service teachers, there are many who come to value the routine and intend to incorporate it into their practice, as demonstrated in the following excerpt from a pre-service teacher:

> Doing this with my table group also made me become more comfortable with the idea of thinking aloud regarding “what do you notice and what do you wonder”? I started to feel less pressure and I get the notion of “no statement is wrong” and what I notice and wonder is still important. As a teacher I will try to reflect this in my classroom so students feel comfortable with the fact that it is not all about just one right answer. (Pre-service teacher reflection)

The notice-wonder routine is about promoting the fundamental habit of mind necessary for doing mathematics and achieving mathematical proficiency – the habit and disposition to notice AND value what one is noticing; the habit and disposition to be curious about what one is noticing and to value that curiosity; the habit and disposition to care about one’s own and others’ noticings and wonderings in a community (Fetter, 2017; Zager, 2018). Rumack and Huinker’s (2019) example in a middle school (Grade 7 or roughly Form 1) classroom involving proportional reasoning illustrates the effectiveness of the notice-wonder routine for adolescents who have often learned to be silent in the mathematics classroom.

The notice-wonder routine is suitable from pre-kindergarten through upper secondary/high school as learners need to be reminded that it is curiosity that drives learning and that there are multiple things one might notice about a phenomenon. In addition, the notice-wonder routine is not one that should be used for mathematics only since the powers of observation and curiosity are needed in all areas of human activity and learning. Table 1 explains how to implement the notice-wonder routine (adapted from Rumack & Huinker, 2019, p. 396).

Table 1

An example of how to implement notice-wonder routines

Select a mathematical prompt	Prompt should lead to mathematical observations and be related to a meaningful mathematical goal (either content or process). The prompt can include short scenarios, numberless word problem, visual image(s), audio clip(s).
Start with the prompt	Show/give the mathematical prompt. Read or describe it if necessary. Ask students "What do you notice"? "What do you wonder"? and give sufficient time for independent thinking, peer discussion. Direct students to write down their noticings (at least three) on the provided sheet or instruct them to draw a table. The teacher may also do the recording on a visible surface if a whole-class activity.
Time to share & listen	Remind students about respectful listening and questioning. Encourage each student to offer what they noticed and wondered about even if already recorded and especially encourage students who are less eager to participate.
Whole-class discussion	Record and chart the noticings and wonderings.
Decision point	Based on the mathematical goal of the lesson, decide which noticings and wonderings to follow.

Routine: Dot talks

Subitizing, or the ability to quickly recognize a small quantity of objects without counting, is a key mathematical skill necessary for beginning to work with quantity arithmetically in the early years (Clements & Sarama, 2014). Difficulties subitizing are associated with dyscalculia – difficulty with mental arithmetic and general arithmetic (Butterworth, 2018). Dot talks are a routine that supports the development of subitizing and, in recent times, have been popularized by Stanford Professor Jo Boaler (https://www.youcubed.org/resources/jo-teaching-visual-dot-card-number-talk/) and others (Parrish, 2014). Though intended for younger learners (PreK -1) they are valuable for learners at all age levels and promote the habits of recognizing and using patterns in numbers for computation.

In a typical dot talk, an image of a series of dots is presented and students are asked how many dots they see. However, the routine does not end there with the right answer; the follow- up prompt is for students to describe or explain how they determined the number of dots. An example is shown below.

Teacher: Without counting, determine how many dots are in the image. Explain how you got that number. [Show Figure 2]

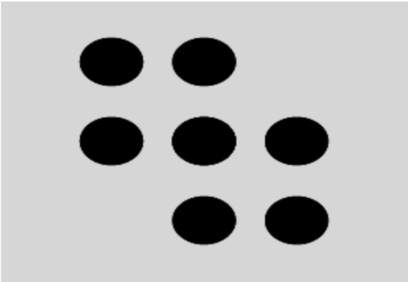

Figure 2
Dot image. (Image credit Steve Wyborney retrieved from http://ntimages.weebly.com/points--dots.html)

Student: There are 7 dots. I saw 2 in the first row, 3 in the second and 2 in the last. 2 + 3 + 2 is 7.

Student: 7. I got the same sum, but I saw 2, 3 and 2 in the columns.

Student: 7. I saw a 5-dot dice pattern in the middle and 2 dots on the ends. 5 + 2 is 7.

Student: I saw a 4 at the top and a 3 below to make 7.

Student: I saw two 4s but they shared 1 dot in common, so I had to minus 1 from 8.

Student: I see a 3 at the top and a 3 at the bottom and a 1 in the middle. 3 +3+1 = 7.

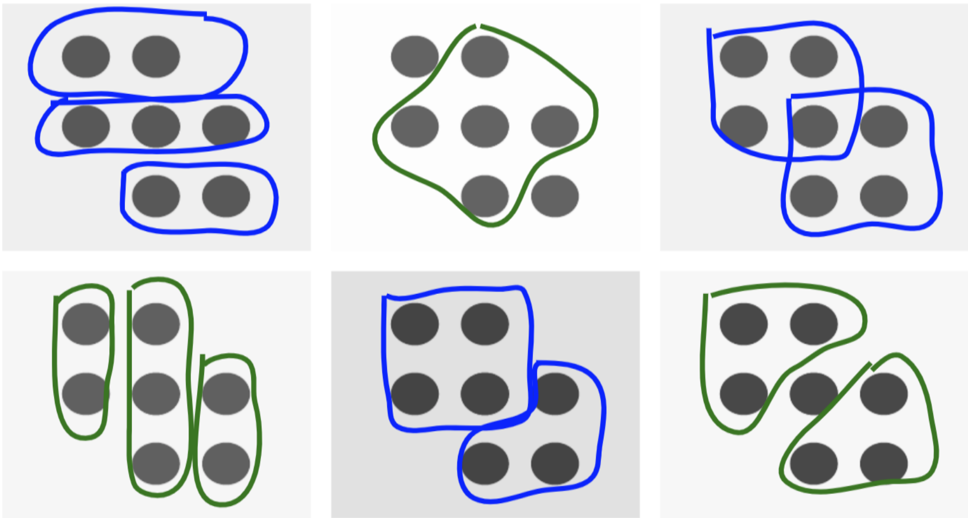

Figure 3
Teacher generated record of student thinking.

In this simple arrangement (Figure 2 & Figure 3), there is opportunity for many students to come to the realization that in mathematics there is more than one way to arrive at the same answer and, as with the WODB routine, the conversational emphasis in presenting their answer becomes less about who is correct and more about *how* each interpretation and differential learner focus is correct in its own way. Further, the presentation allows each learner to make sense of and have some practice with a variety of simple addition sums (and in one case subtraction). Seeing this sort of engagement and having this opportunity regularly increase the occasions for students who struggle with subitizing to practise and have their growth recognized alongside that of others.

In terms of connecting to more advanced problems, I leave it to readers to consider how the aforementioned strategies might come into play in this problem adapted from a primary/elementary level mathematics contest (Figure 4) or another based around a particular modification of the dot talk "#dice chats" found on Facebook (Figure 5).

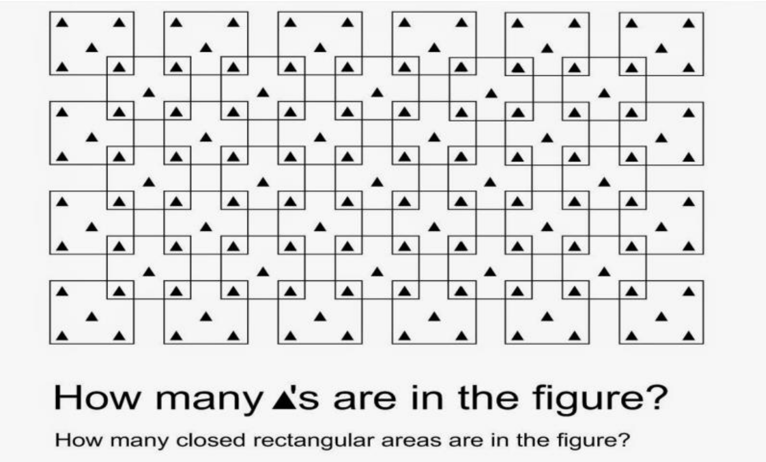

Figure 4
Adapted mathematics contest problem (Solution left as an exercise for readers)

Figure 5
Dice chat image (image credit Adam Hillman https://pikdo.net/p/witenry/1782255366789574480_1210061824)

Number talk (arithmetic) routines

Dot talks in the earlier grades that develop the ability to subitize give way to number talks in the later grades. [Note: To the extent that dot talks develop the quantity sense of number, a dot talk *is* a number talk.] A number talk is a short routine, typically no more than 10 minutes, focused on a single problem that is amenable to multiple strategies and approaches. As with dot talks and WODB, the teacher's role shifts from being the "expert" to recording and assisting students to make connections among strategies presented (Parrish, 2014) and pressing students to explain and communicate clearly "how they know". An example script is illustrated in Figure 6. The age group for the conversation will depend on the specific curriculum in one's jurisdiction as well as the specific competencies and proficiencies learners are expected to develop. However, the intention here is to demonstrate the routine in a general sense, not to prescribe teaching for a specific age grouping, given the diversity of what learners within any age cohort know.

Teacher: Our number talk problem for today is 17 + 18. Solve the problem using mental arithmetic but don't shout out the answer. Think about how you will explain how you solved the problem.

[Time for students to do the mental computation and think about their strategy.]

Student: I got 35. I broke up the numbers into 10 and 7 and 10 and 8. I added the two tens to get 20 and the 7 and 8 to get 15. I added 20 and 15 to get 35.

Student: I got 35 too but I used doubles. I know 17 and 17 is 34 and I had one left over from the 18.

Student: I broke up the 18 into 3 and 15. I added the 3 to 17 to get 20 and the left-over 15 to get 35. I suppose I could have broken up the 17 into 2 and 15 too and worked with the 18.

Student: I see another way. Break up both numbers into 15s. The double of 15 is 30 and the remaining 2 and 3 make 5.

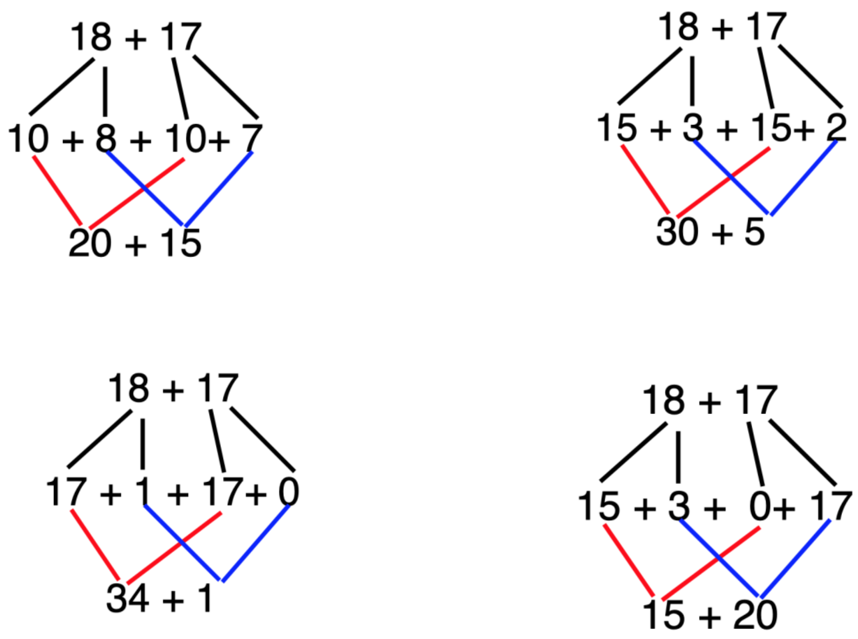

Figure 6
Teacher's recording of student narrated strategies (L to R and T to B: place value decomposition, doubles, doubles and double plus 1, decomposition using benchmark numbers, and making a ten)

In the example in Figure 6, note that all students arrive at the same answer BUT the value of the routine is in all students having an opportunity to value other approaches that are equally valid. Instead of a single "right way" of doing the problem, students get to use – or see used and explained – approaches that involve whole number decomposition and recomposition, including the use of number pairs and base-ten decomposition. Note as well that in this example, none of the students used "the standard algorithm" as this is not well-suited to mental arithmetic, but gained its foothold during the period of mercantilist trade in the 15th and 16th centuries with the need for clear accounting ledgers showing what we now call basic arithmetic (Harouni, 2015).

The problem could also be approached using a compensation subtraction method (e.g., 20+18-3). To be clear, not every problem students work on should be done as a number talk. Rather, teachers should develop the routine in order for students to become comfortable with mental arithmetic and be able to explain their thinking as well as to have repeated opportunities to see and value the diverse strategies used by their peers and to make connections amongst these. Work with number talks in classrooms develops the necessary flexibility with computation that is a large part of number sense fluency (Shumway, 2011).

Routine: Pattern talks

While significant attention has been given to arithmetic routines – a consequence of many curricula having an overrepresentation of this domain for early learners – helping learners to make the difficult transition from arithmetic thinking to algebraic thinking is necessary. The visual patterns website http://www.visualpatterns.org provides many interesting problems in which the first three terms of a sequence are presented visually with a prompt to find the 43rd term (or some other number). Figure 7 is adapted from pattern #241 (credited to Elizabeth Kelley) and includes a differentiated variant to be used if students have difficulty seeing the "constant" and "variable" parts of the pattern.

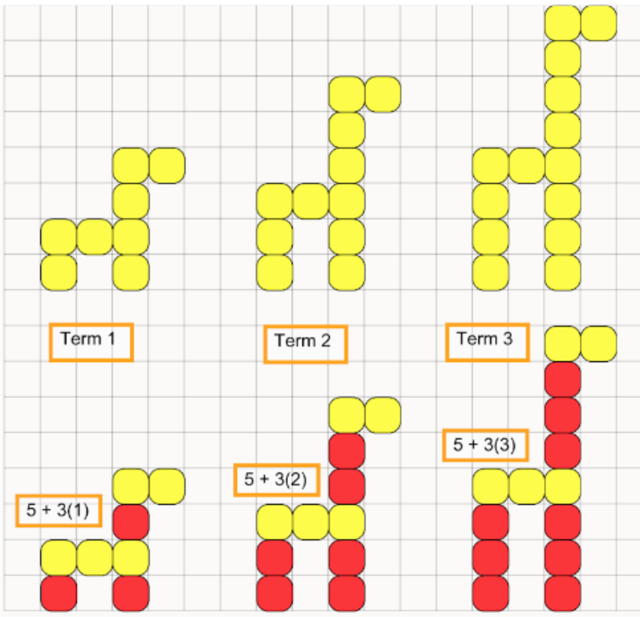

Figure 7
Giraffe pattern (image credit Steven Khan)

In working with this problem in the past, learners at the beginning stage of the arithmetic-to-algebraic transition can typically describe how the pattern is growing ("the head and body are not changing but the legs and neck are changing by 1 each time"), but have difficulty extending the pattern/generalization and in using algebraic notation. As a routine, the conversations in a pattern talk have a different focus, viz. identifying the constant parts of a pattern, the variable part of the pattern and representing that change symbolically (often in more than one way), and helping learners to make the discernment that different representations of a pattern show attention to different aspects but are equivalent representations. This realization is possible also using several different images on the visual patterns website (e.g., several of the patterns are of the triangular number sequence).

Routine: Mobiles

Another routine for developing aspects of algebraic thinking are mobiles, which demonstrate the core concept of equality as a balance as opposed to equivalence of representations with the visual pattern example described previously and represented in Figure 7. In Figure 8, the mobile is balanced and the sum of all the parts is 96. Students must work out the value of each shape. As with other routines, there can be more than one way to get there, though in this case the problem structure suggests starting with the (pink) trapezium, 6 of which must be half of 96 – or 48 – thus giving its value as 8. On the left-hand side of the puzzle, the pink trapezium is placed in the middle of the balance, which means it does not affect the balance of the squares and hearts but does affect the total. It is an interesting variation that generates some conversation about the meaning of equality.

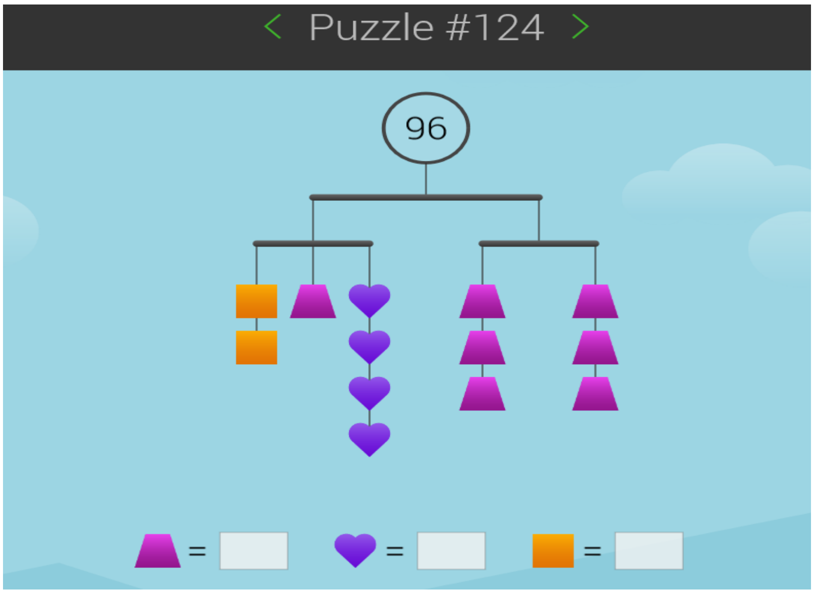

Figure 8
SolveMe Mobile Puzzle number 124 (https://solveme.edc.org/mobiles/).

Routine: Missing Numbers Patterns

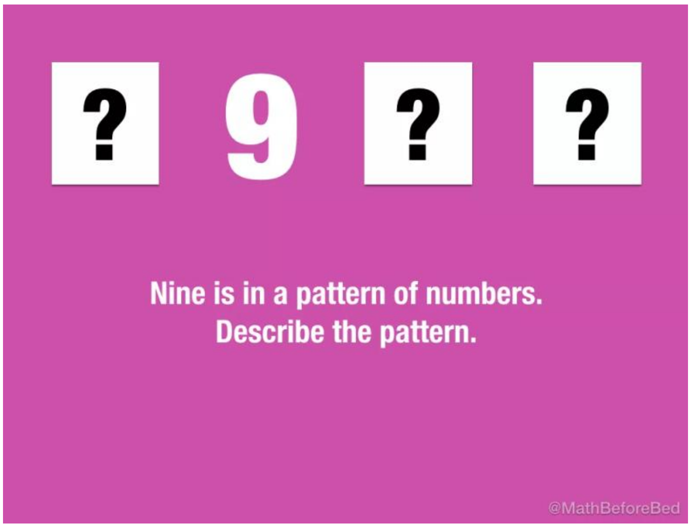

Figure 9
A complete-the-pattern routine (Source: https://mathbeforebed.com/2017/06/09/a-pattern-around-nine/.

This routine (Figure 9) is a variant of single-answer problems that require students to discern a single pattern and find the missing number, such as those in the visual pattern example presented previously (Figure 7). This routine, however, invites students to create their own pattern and use their own pattern rules, thus giving them agency and authority over their ability to create and do mathematics. In use, individual students can present their patterns and others can work on figuring out the pattern rule or finding another term (say the 10^{th}) or writing the rule in an explicit form.

Routine: Quick draw and quick builds

Another area of the mathematics curriculum in the early years that is under-represented and that likely gets insufficient practice in the Caribbean and elsewhere is that of visual and spatial reasoning that builds towards later geometric thinking. The quick draw routine (also Can you Draw This https://wordpress.oise.utoronto.ca/robertson/portfolio-item/can-you-draw-this/) engages learners in paying attention to geometric properties of figures and representing them visually. The quick draw or quick build routine involves flashing an image of a drawing or geometric construction (usually with pattern blocks or connecting cubes or another geometric manipulative such as a geoboard). Students are then given a period to draw or build on their own, and then either having a discussion or delaying the discussion until after a second flash of the image and giving students another opportunity to adjust their drawing/construction. Such routine tasks develop students' capacity to notice and consciously attend to aspects of geometric and spatial information (e.g., symmetry, orientation), to geometrically "chunk" and use visual strategies such as symmetry, and to actively use vocabulary in the discussion about how they remembered and what information they used. A sample script follows and includes a quick image/draw (Figure 10). Figures 11 and 12 are additional examples that can be used with pattern blocks (2D area filling) and interlocking cubes (3D spatial visualization).

Teacher: I will display an image for about five seconds. Look at the image and try to remember what it looks like. On your paper/mini white-board draw what you remember (use the blocks to build it). For harder figures I may show it more than once and you can modify your drawing/construction. We will have a "geometry talk" – describe how you thought about it, what helped you remember.

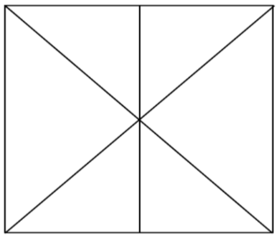

Figure 10
Quick image/draw (Source: https://wordpress.oise.utoronto.ca/robertson/files/2016/11/Can-You-Draw-This-BLM-Dec-2016.pdf).

Student: I saw a square with lines running from each corner to the opposite corner. When you showed the image a second time I saw another line running down through the centre from the top to the bottom.

Student: I saw it like two hourglass shapes made of triangles, one straight up and one turned sideways. It reminded me of something I saw on the computer.

Student: I saw an X with a line through the middle of it and then I saw the box that made it a square.

Student: I saw four small triangles – two on top and two below and two larger triangles on the side. They all looked like right-angle triangles. [This would be an interesting path for the teacher to follow.]

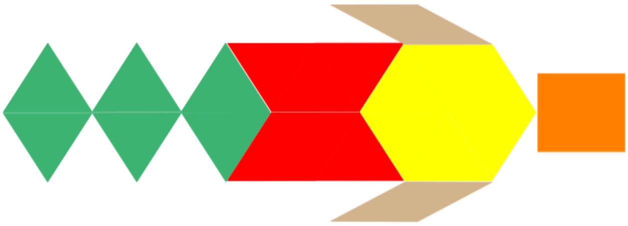

Figure 11
Pattern block quick image/quick build (Students sometimes call this the mermaid/aquaman. Source: *https://wordpress.oise.utoronto.ca/robertson/files/2017/03/Pattern-Blocks-Quick-Image-Slides.pdf*

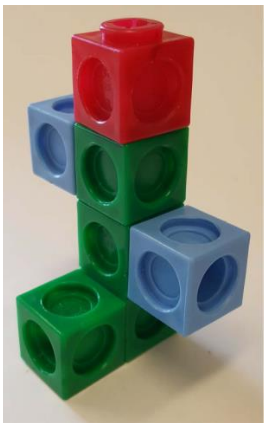

Figure 12
Quick build image (Source: *https://wordpress.oise.utoronto.ca/robertson/files/2017/03/Multilink-Cube-3D-Quick-Image-Challenges-Slides.pdf*

Parent math routines

Although not a main focus of this paper, there is increasing emphasis among high-performing school jurisdictions (regardless of community socio-economic status) to partner more meaningfully with parents in their children's education (Kreisberg & Beyravesnand, 2019; Philipson et al., 2017). Alongside this phenomenon are a number of publications and associated websites (e.g. http://bedtimemath.org/; https://mathbeforebed.com/) marketed to parents that seek to make mathematical conversation a routine part of parent-child interaction. In addition, an increasing number of games with an explicit mathematical focus (usually arithmetic, number or geometric) have made it to market in recent years (e.g., https://primeclimbgame.com). A discussion of these is outside the scope of this paper, though, as with routines in general, there is a need for empirical research in the Caribbean around the interactions of parents and children in settings outside of school around mathematics (Hanner et al., 2019).

How Routines Support Meaningful Problem Solving: The Finger-Counting Problem

In this section I demonstrate how mathematical routines that have been developed with pre-service teachers (or students) come into play in the context of a problem-solving task – the finger counting problem, illustrated in Figure 13. A sample lesson script follows.

Teacher: Hold your left hand in front of your face with palm facing towards you (thumb on the left). Count and tap each finger starting from 1 on the thumb. When you get to the pinky (little-finger), turn around and continue the counting sequence. When you get to the thumb turn around again. Keep repeating this movement. Which finger are you on when you say the number 1,000. [Extension: Are you moving towards or away from the thumb when you say the number 1,000? How do you know?] (Adapted from Collaborative Mathematics, n.d. http://www.collaborativemathematics.org/challenge03.html)

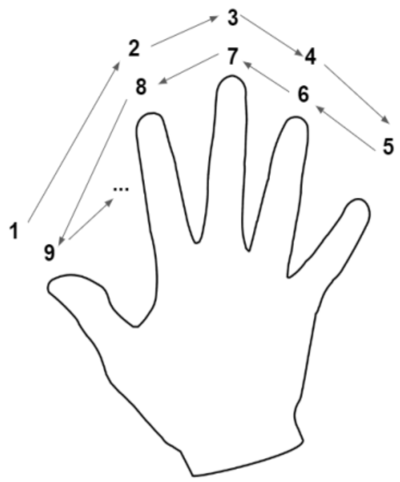

Figure 13
Illustration of the finger counting problem.

In this lesson, individual skip counting or whole-class choral skip counting routines are practised. If properly recorded, it is possible for the skip counting by 8s to be available as a "clue" for learners in recognizing the skip counting by 8s patterns. Attention should not be drawn to this, however, before the problem is attempted. The notice-wonder routine also supports the necessary engagement in this problem. The problem solvers have to *notice* that the finger-to-number mapping is a one-to-many relation and must *wonder* about ways to represent this mapping in a way that opens up the problem. Many draw a direct representation of a hand and begin to place the numbers on each finger in order to look for patterns. As they work on the problem, there are opportunities to notice more things, such as the fact that the even numbers (index and ring) and odd numbers (thumb, middle, and pinky) occur on specific fingers, which leads to a conjecture that the number 1,000 must lie on one of these fingers. At this point, students will wonder how to determine which of these two fingers is the right one without having to count all the way to 1,000.

This is the period of productive struggle (Warshauer, 2015). The moves from this point depend on what other routines students have worked with or what tools they are familiar with and have access to in their classroom. For example, some students may use a number path or hundreds grid in an attempt to identify other patterns that could help. Some may choose to use tables, (modulo) clock diagrams, and others, algebraic thinking and representations depending on the grade level. A complete lesson plan used with pre-service elementary teachers for this lesson can be found at the link https://tinyurl.com/yyyn6mc8.

In the sessions leading up to this problem, the pre-service teachers engage in choral counting (skip counting) routines as described in Franke et al. (2018) – though without the daily/weekly frequency one would employ in a regular classroom – and have done work with patterning including visual patterns and finding missing number patterns. The key to the problem, however, is in using the skills and competencies in number sense and algebraic thinking as well as noticing and wondering to pose the question, “How do the numbers on each finger form a pattern”? Using different tools or representations (e.g., colour) students have to discern that there is an 8 cycle (modulo 8); in other words, the pattern restarts every 8th number. From this, the question/wonder of what is the relation of 1,000 to 8 must be posed and answered.

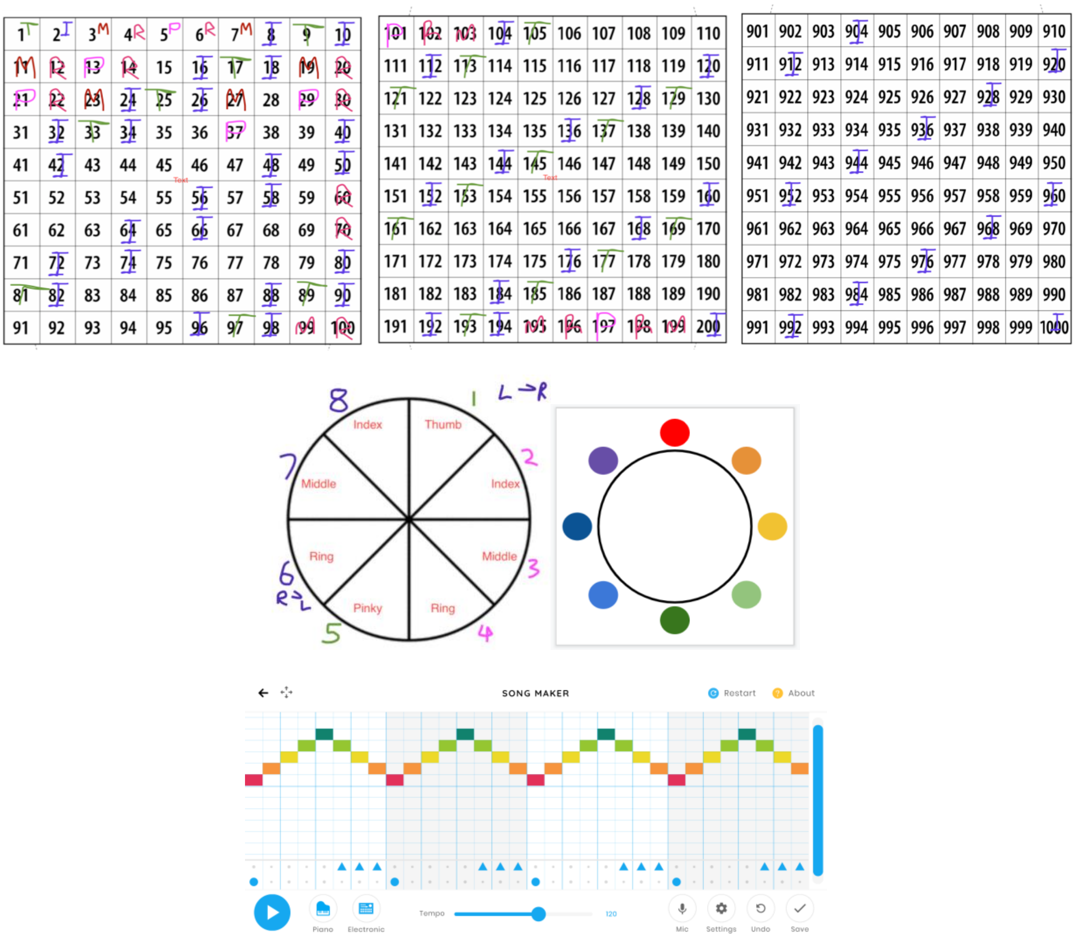

Figure 14

Thousand chart, clock diagram, colour wheel, and sonic representations of problem that leads to solution. (For audio: https://musiclab.chromeexperiments.com/Song-Maker/song/6059102348247040).

Relationships Build Reform

Middleton et al. (2002) presciently argued that relationships at multiple levels are the key drivers of change in complex learning systems (Davis & Simmt, 2003), and it is on this note I wish to end. While the purpose of math routines can be seen within a traditional utilitarian perspective on increasing mathematics achievement, the perspective I wish to advance is that the type of relationships presented in this paper, as well as their discursive entailments, offer an opportunity to break with traditional views of mathematics teaching and learning and increase the likelihood of more meaningful relationships and engagement for more diverse learners emerging in mathematics classrooms in the Caribbean.

The routines I have presented and the short sample scripts offer possibilities for decentering the teacher as the sole authority or repository of mathematics knowledge in the classroom and for learners to notice that all students in the class can be authorities on their knowledge and ways of knowing. In classrooms where I have observed these routines and where the pre-service teacher candidates I have worked with have tried them, there is something akin to joy. Some of this is related to the aesthetics of the materials. But a significant part of it is related to sharing with each other their understanding and appreciation of diverse ways of seeing a problem. For teachers, the routines offer a moment when they can step into the role of actively listening and interpreting students' conceptual understanding.

CONCLUSION

I could easily continue to fill multiple pages with descriptions and examples of powerful routines for mathematics instruction that contribute to mathematical proficiency and flourishing. The list of references, books, and websites that is included in the paper and reference list are good places to start, as is the Mathematics Twitter Blog-o-sphere (#MtBoS). But my purpose in this paper has been to illustrate the necessity of mathematically powerful routines that support ambitious pedagogy and develop students as resilient mathematical problem solvers. My own experience in schools in Trinidad and Canada as a learner, teacher, and now researcher suggests that we undervalue these powerful moments of meaning- and sense-making for students and provide insufficient opportunities for all students to engage and demonstrate their developing mathematical proficiencies and to build resilient, compassionate classroom communities. Routines such as those presented here (in contrast to cognitively low demand and discursively diminutive tasks) offer opportunities for everyone to participate and experience success on a very regular basis in relatively low-stakes learning situations. If our goal is to have students develop proficiency and enjoy doing mathematics with others, mathematical routines need to become a valued and regular part of the classroom teacher's repertoire.

As mentioned earlier, there is a need for specific attention to be given to instructional (process) and mathematical (task) routines in Caribbean settings. Although the idea of routines may seem trivial, some fraction of the persistent achievement gap might be reduced through the intentional focus on the high learning ratio afforded by these or other routines. I look forward to reading the many action research projects inspired by a careful consideration of the potential value of leveraging powerful routines more "routinely" and intentionally.

I end this paper by inviting Caribbean mathematics educators at all levels to adapt, remix, and share their own powerful contextual and culturally relevant routines for learning mathematics in their local networks and via social networks (both digital and peer). In particular, I want to strongly encourage groups of teachers, heads of department, administrators, and parents to come together to plan and practise these routines so that students' experiences of mathematics in school and in the world are more coherent and intentional. When sharing good practice in a generous, careful, and contextualized manner becomes a "routine" part of the work of teaching and learning mathematics in Caribbean classrooms and jurisdictions, we can expect to see large gains for a greater number of students along multiple dimensions as well as increased enjoyment.

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